MATH 121A Prep: Eigenvalues

1. Find all eigenvalues of the matrix $\begin{bmatrix} -2 & 0 & 0 \\ 1 & 2 & 5 \\ 4 & 2 & -1 \end{bmatrix}$.

Solution: To find the eigenvalue we need to find the zeros of the polynomial:

$$\det(A-\lambda I_3) = \begin{vmatrix} -2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 5 \\ 4 & 2 & -1-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} 2-\lambda & 5 \\ 2 & -1-\lambda \end{vmatrix} = (-2-\lambda)((2-\lambda)(-1-\lambda)-10)$$

$$= (-2-\lambda)(-2-\lambda+\lambda^2-10) = (-2-\lambda)(\lambda^2-\lambda-12) = (-2-\lambda)(\lambda-4)(\lambda+3)$$

So the eigenvalues are $-2, 4, -3$.

2. Find all eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 4 & 5 \\ 1 & 8 \end{bmatrix}$.

Solution: The eigenvalues are the zeros of:

$$\det(A-\lambda I_2) = \begin{vmatrix} 4-\lambda & 5 \\ 1 & 8-\lambda \end{vmatrix} = (4-\lambda)(8-\lambda)-5 = 32-12\lambda+\lambda^2-5 = \lambda^2-12\lambda+27 = (\lambda-3)(\lambda-9)$$

So the eigenvalues are 3 and 9.

$\lambda = 3$:

$$(A-\lambda I_2)\vec{v} = \vec{0} \to \begin{bmatrix} 1 & 5 & 0 \\ 1 & 5 & 0 \end{bmatrix} \xrightarrow{R2=R2-R1} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This means x_2 is a free variable and $x_1 = -5x_2$. Thus solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix}$. Then $\begin{bmatrix} -5 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 3$.

$\lambda = 9$:

$$(A-\lambda I_2)\vec{v} = \vec{0} \to \begin{bmatrix} -5 & 5 & 0 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{R1=R1+5R1} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

This means x_2 is a free variable and $x_1 = x_2$. Thus we have solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 9$.

3. Recall that we defined the eigenspace of an $n \times n$ matrix A corresponding to an eigenvalue λ as the set of all eigenvectors corresponding to λ as well as the zero vector. We can also write this as $V = \{\vec{v} : A\vec{v} = \lambda\vec{v}\}$ Prove that this is a subspace of $\mathbb{R}^n$.

Solution: Non-empty: The zero vector is in the eigenspace by definition.

Closed Under Addition: Suppose $\vec{v_1}$ and $\vec{v_2}$ are in V. Then we have

$$A(\vec{v_1} + \vec{v_2}) = A\vec{v_1} + A\vec{v_2} = \lambda\vec{v_1} + \lambda\vec{v_2} = \lambda(\vec{v_1} + \vec{v_2})$$

so $\vec{v_1} + \vec{v_2} \in V$.

Closed Under Scalar Multiplication: Let $\vec{v} \in V$ and c a real number. So,

$$A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v} = \lambda(c\vec{v})$$

so $c\vec{v}$ is an element of V as well.

Therefore the eigenspace is a subspace.